Assignment 10.

This homework is due *Tuesday* 12/07/2010.

There are total of 22 points in this assignment. 19 points is considered 100%. If you go over 19 points, you will get over 100% for this homework and it will count towards your course grade.

Collaboration is welcome. If you do collaborate, make sure to write/type your own paper.

This assignment covers section 7.1.

(1) [3pt] (Exercise 7.1.8) If $f \in \mathcal{R}[a,b]$ and $|f(x)| \le M$ for all $x \in [a,b]$, show that

$$\left|\int_a^b f\right| \le M(b-a).$$

(2) (a) [3pt] (Exercise 7.1.9) If $f \in \mathcal{R}[a,b]$ and if $(\dot{\mathcal{P}}_n)$ is any sequence of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$, prove that $\int_a^b f = \lim_{n\to\infty} S(f;\dot{\mathcal{P}}_n)$.

(b) [3pt] (Exercise 7.1.10) Let $g(x) = 0$ if $x \in [0,1]$ is rational and $g(x) = 1/x$ if $x \in [0,1]$ is irrational. Prove that $g \notin \mathcal{R}[0,1]$. However, show that there exists a sequence $(\dot{\mathcal{P}}_n)$ of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ as $n \to \infty$ and $\lim_{n\to\infty} S(g;\dot{\mathcal{P}}_n)$ exists.

(3) (a) [3pt] (Exercise 7.1.11) Suppose that f is bounded on $[a,b]$ and that there are two sequences of tagged partitions of $[a,b]$ such that $\|\dot{\mathcal{P}}_n\| \to 0$ and $\|\dot{\mathcal{Q}}_n\| \to 0$ as $n \to \infty$, but such that

$$\lim_{n\to\infty} S(f;\dot{\mathcal{P}}_n) \ne \lim_{n\to\infty} S(f;\dot{\mathcal{Q}}_n).$$

Show that f is not in $\mathcal{R}[a,b]$.

(b) [3pt] (Exercise 7.1.12) Consider the Dirichlet function, defined by $f(x) = 1$ for $x \in [0,1]$ rational and $f(x) = 0$ for $x \in [0,1]$ irrational. Show that f is not Riemann integrable on $[0,1]$. (Hint: You can use problem 3a.)

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(4) (a) [4pt] (Exercise 7.1.13) Suppose that $f : [a, b] \to \mathbb{R}$ and that $f(x) = 0$ except for a finite number of points $c_1, \ldots, c_n$ in $[a, b]$. Prove that $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = 0$. (Hint: In a given partition, how many intervals may have a tag with nonzero value of f?)

(b) [3pt] (Exercise 7.1.14) If $g \in \mathcal{R}[a, b]$ and if $f(x) = g(x)$ except for a finite number of points in $[a, b]$, prove that $f \in \mathcal{R}[a, b]$ and that $\int_a^b f = \int_a^b g$. (Hint: You can use problem 3b applied to $f - g$.)